Intersymbol Interference-Free Nyquist Pulse Shaping Using Quadratic Bézier Curves

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***Abstract*—This paper introduces a novel family of intersymbol interference (ISI)-free Nyquist pulses constructed using quadratic Bézier curves. In contrast to conventional pulse shaping approaches that rely on fixed trigonometric or polynomial basis functions, the proposed design methodology exploits the inherent geometric flexibility of Bézier curves to precisely control the frequency response roll-off characteristics. This framework introduces a tunable control point whose coordinates are systematically optimized to minimize the bit error rate (BER) under conditions of symbol timing uncertainty. Comprehensive numerical evaluations demonstrate that the resulting Optimal Bézier Pulse (OBP) achieves superior performance compared to the classical raised-cosine pulse and other state-of-the-art parametric pulse designs across a wide range of roll-off factors, exhibiting enhanced robustness against timing jitter impairments.**

***Index Terms*—Bézier curves, intersymbol interference (ISI), Nyquist pulses, pulse shaping, timing jitter.**

I. Introduction

The raised-cosine (RC) pulse, originally introduced in the seminal work by Nyquist [1], remains one of the most extensively employed pulse shaping filters in digital communication systems susceptible to intersymbol interference (ISI). Fundamentally, the RC pulse operates as a low-pass filter characterized by a cosine-shaped spectral roll-off region, which facilitates compliance with the Nyquist criterion for ISI-free transmission.

Notwithstanding its widespread adoption, the RC pulse does not represent the optimal solution in terms of error performance, particularly under imperfect timing synchronization conditions. Consequently, substantial research effort has been devoted over the past several decades toward the development of alternative Nyquist pulse designs exhibiting enhanced robustness to timing errors [2]–[11].

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A significant advancement was presented in [2], where the authors introduced the *better than raised cosine* (BTRC) pulse. This design demonstrated measurable improvements over the conventional RC pulse with respect to maximum distortion, eye diagram aperture, and probability of error in the presence of symbol timing offsets. Subsequent investigations extended this work by proposing Nyquist pulse families that surpass the BTRC performance, albeit frequently lacking closed-form time-domain representations [3]–[5].

Specifically, the work in [3] introduced pulse shaping functions based on either the *flipped-hyperbolic secant function* (fsech) or its inverse (farcsech). Furthermore, the authors in [4], [5] developed comprehensive families of ISI-free Nyquist pulses utilizing composite inner and outer function constructions. Among these designs, the pulse employing the *inverse cosine* as the outer function and the *logarithm* as the inner function (denoted acos[log]) exhibited performance superior to all previously documented pulses across various combinations of the roll-off factor α and timing jitter τ. In our previous work [6], we propose a new two-parametric ISI-free Nyquist pulse based on the composition of inverse hyperbolic cosecant and secant functions (acsch[asech]). This pulse depends solely on the roll-off factor and the timing jitter parameter. Performance analysis demonstrates that it achieves superior side-lobe suppression and wider eye-diagrams compared to state-of-the-art pulses like acos[log] and acos[asinh].

A parallel research direction has focused on parametric Nyquist pulse designs, as investigated in [7]–[12]. The approach in [7] employed a fourth-degree polynomial formulation wherein the polynomial coefficients serve as adjustable design parameters. Additionally, the works in [8]–[12] introduced the concept of piecewise Nyquist pulse construction, whereby the frequency response is partitioned into distinct sub-regions, each characterized by an independent polynomial function.

While these parametric approaches yield competitive or improved performance relative to conventional pulses, they typically necessitate the determination of optimal parameter values for each specific combination of roll-off factor and timing jitter. This requirement presents practical challenges, as timing jitter in operational communication systems arises from stochastic synchronization impairments and cannot be predetermined as a fixed design parameter, unlike the roll-off factor which may be specified a priori.

The principal motivation of this work is to propose a novel ISI-free Nyquist pulse that achieves superior performance across multiple performance metrics, including maximum distortion, eye diagram opening, and error probability, without requiring complex parameter reconfiguration for varying jitter conditions. To accomplish this objective, we employ quadratic Bézier curves to shape the pulse frequency response. The proposed pulse emerges from a systematic optimization procedure, wherein the optimal control point coordinates, once determined, yield a pulse exhibiting superior performance for essentially any combination of roll-off factor and timing jitter values within the practical range of interest.

II. ISI-Free Bézier Nyquist Pulse

A. Fundamental Concepts

A Bézier curve constitutes a parametric curve whose general mathematical formulation is given by

$$p(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i}\, t^i\, p_i, \tag{1}$$

where $p(t)$ denotes the coordinates of the curve, the parameter t satisfies $0 \leq t \leq 1$, $\binom{n}{i}$ represents the binomial coefficients, and $p_0, p_1, \ldots, p_n$ constitute the control points. The curve originates at point p_0 and terminates at point p_n, while the intermediate control points $p_1, p_2, \ldots, p_{n-1}$ influence the curve shape without lying on the curve itself.

For the degenerate case $n = 1$, the formulation reduces to a *linear* Bézier curve, which corresponds to a straight line segment connecting points p_0 and p_1. For $n = 2$, we obtain the *quadratic* Bézier curve:

$$p(t) = (1-t)^2\, p_0 + 2(1-t)\, t\, p_1 + t^2\, p_2, \tag{2}$$

where the parameter t satisfies $0 \leq t \leq 1$. The derivative of (2) with respect to t yields

$$\frac{d}{dt}\, p(t) = 2(t-1)\,(p_1 - p_0) + 2t\,(p_2 - p_1), \tag{3}$$

which demonstrates that the tangent vectors to the curve at the endpoints p_0 and p_2 intersect at the intermediate control point p_1.

Fig. 1 illustrates the fundamental concept underlying the proposed Bézier Nyquist pulse construction. The starting and ending control points are fixed at $(B(1-\alpha), 1)$ and $(B, 1/2)$, respectively, where B denotes the Nyquist bandwidth and α represents the roll-off factor. The intermediate control point p_1 with coordinates (x_1, y_1) may be positioned at any location in the Cartesian plane and serves as the primary design parameter.

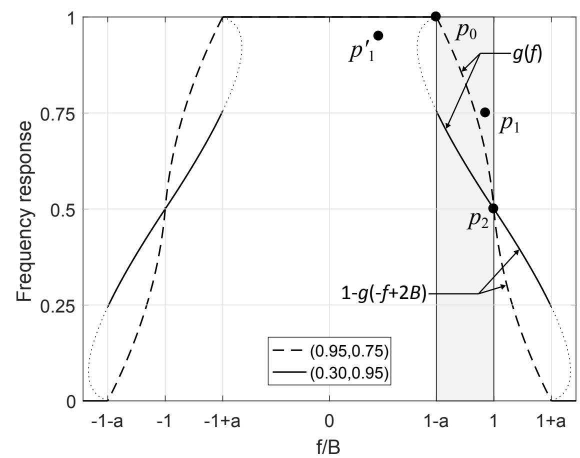

Fig. 1. Frequency response of the proposed Bézier Nyquist pulse for control point coordinates $p_1 = (0.95, 0.75)$ and $p'_1 = (0.3, 0.95)$, demonstrating continuous and discontinuous roll-off characteristics, respectively.

The frequency response of the proposed Bézier Nyquist pulse is defined as

$$S(f) = \begin{cases} 1, & |f| \leq B(1-\alpha), \\ g(f), & B(1-\alpha) < |f| < B, \\ 1/2, & |f| = B, \\ 1 - g(-f+2B), & B < |f| < B(1+\alpha), \\ 0, & |f| \geq B(1+\alpha), \end{cases} \tag{4}$$

where the parametric functions are given by

$$f(t) = (1-t)^2 B(1-\alpha) + 2(1-t)t\, x_1 + t^2 B, \tag{5a}$$

$$g(t) = (1-t)^2 + 2(1-t)t\, y_1 + t^2 \cdot \frac{1}{2}, \tag{5b}$$

with the parameter t constrained to $0 \leq t \leq 1$.

The term $1 - g(-f + 2B)$ represents a 180° clockwise rotation of the function $g(f)$ about the point $(B, 1/2)$. Equivalently, the portion of the Bézier Nyquist pulse for $B < f < B(1+\alpha)$ can be constructed using (2) with the transformed control points $p_0 = (B, 1/2)$, $p_2 = (B(1+\alpha), 0)$, and $p_1 = (2B - x_1, 1 - y_1)$. Furthermore, the pulse exhibits even symmetry about the origin, which permits the analysis to focus exclusively on the function $g(f(t))$ within the positive frequency domain.

Based on the quadratic Bézier formulation in (2), the parametric frequency function $f(t)$ may extend beyond the interval $[B(1-\alpha), B]$ as the parameter t varies from 0 to 1. However, the pulse definition in (4) stipulates that the function $g(f(t))$ is valid only when $B(1-\alpha) < f(t) < B$. Consequently, this discrepancy may induce a discontinuity in the frequency response. For instance, Fig. 1 illustrates $S(f)$ for the control points $p_1 = (0.95, 0.75)$ and $p'_1 =$

TABLE I
BIT ERROR RATE (BER) PERFORMANCE COMPARISON WITH $N = 2^9$ INTERFERING SYMBOLS AND SNR = 15 dB

α	Pulse Type	$t/T = \pm 0.05$	$t/T = \pm 0.1$	$t/T = \pm 0.2$	$t/T = \pm 0.3$
0.2	acos[log] [5]	6.7242e-8	1.8026e-6	5.5498e-4	1.9027e-2
	acos[asinh] [5]	6.5145e-8	1.6861e-6	5.0950e-4	1.8015e-2
	acsch[asech] [6]	6.2568e-8	1.5439e-6	4.5393e-4	1.6764e-2
	Proposed OBP	**5.6454e-08**	**1.3212e-06**	**3.8838e-04**	**1.4331e-02**
0.25	acos[log] [5]	5.3332e-8	1.0726e-6	2.7416e-4	1.2183e-2
	acos[asinh] [5]	5.1488e-8	9.9816e-7	2.4946e-4	1.1349e-2
	acsch[asech] [6]	4.9210e-8	9.0533e-7	2.1816e-4	1.0306e-2
	Proposed OBP	**4.4623e-08**	**7.9515e-07**	**1.9875e-04**	**9.1089e-03**
0.35	acos[log] [5]	3.5470e-8	4.3365e-7	7.3486e-5	4.5509e-3
	acos[asinh] [5]	3.4124e-8	4.0410e-7	6.7653e-5	4.2252e-3
	acsch[asech] [6]	3.2414e-8	3.6393e-7	5.8702e-5	3.7520e-3
	Proposed OBP	**3.0059e-08**	**3.3382e-07**	**5.8651e-05**	**3.6777e-03**
0.5	acos[log] [5]	2.1559e-8	1.4514e-7	1.4987e-5	1.2082e-3
	acos[asinh] [5]	2.0758e-8	1.3617e-7	1.4609e-5	1.2202e-3
	acsch[asech] [6]	1.9693e-8	1.2137e-7	1.2947e-5	1.1507e-3
	Proposed OBP	**1.9678e-08**	**1.2028e-07**	**1.2517e-05**	**1.1076e-03**

$(0.3, 0.95)$. First, it is observed that the control point is unconstrained and not limited to the interval $B(1-\alpha) < f(t) < B$. This flexibility renders the Bézier formulation a strong candidate for optimizing the impulse shape with respect to BER. Second, the figure demonstrates that while $S(f)$ represents a continuous function for p_1, it exhibits a discontinuity for p_1'.

It is important to recognize that discontinuity in the frequency domain does not preclude continuity in the time domain. A classical example is the Fourier transform relationship between the discontinuous rectangular function and the continuous sinc function. Additionally, since the proposed Nyquist pulse is formulated as a parametric function, we deliberately employ an unconstrained optimization procedure to maximize design flexibility and thereby identify the solution yielding optimal performance, as elaborated in the subsequent section.

The impulse response of the Bézier Nyquist pulse is depicted in Fig. 2 for the control point configurations $p_1 = (0.95, 0.75)$ and $p_1' = (0.3, 0.95)$. Due to the absence of closed-form analytical expressions for these impulse responses, numerical computation via the inverse Fourier transform is employed.

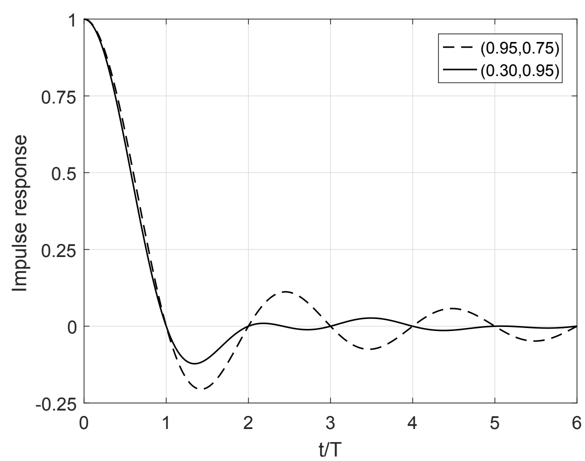

Fig. 2. Impulse response of the proposed Bézier Nyquist pulse for control point coordinates $p_1 = (0.95, 0.75)$ and $p_1' = (0.3, 0.95)$.

B. Optimization-Based Design

Although parametric pulses, such as those proposed in [8]–[12], offer the potential for enhanced performance through optimization of their parameters for specific combinations of roll-off factor and timing jitter, this approach presents practical limitations. In real-world communication systems, timing jitter is inherently a random variable stemming from synchronization uncertainties between transmitter and receiver, rendering precise prediction and pre-optimization infeasible. While the roll-off factor can be fixed during design, systems often demand flexibility to adjust it dynamically based on varying channel conditions or link requirements. Consequently, a more practical strategy is to develop pulses with a fixed shape, such as the proposed OBP, which is optimized once to deliver superior BER performance across a broad range of roll-off factors and timing jitters without necessitating reconfiguration for each scenario.

The average bit error rate (BER) in the presence of ISI represents the definitive performance metric, as it comprehensively captures the combined effects of additive noise, synchronization impairments, and signal distortion [13]. The BER can be computed analytically for binary antipodal signaling following the methodology established in [14].

The primary objective of this work is to develop a novel ISI-free Nyquist pulse that exhibits robust performance su-

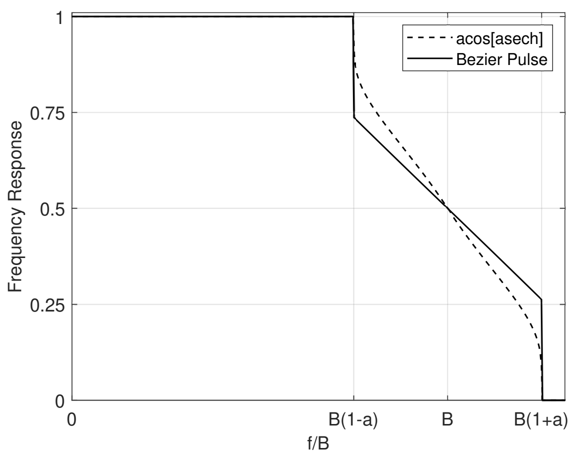

Fig. 3. Frequency responses of the considered Nyquist pulses for roll-off factor $\alpha = 0.25$.

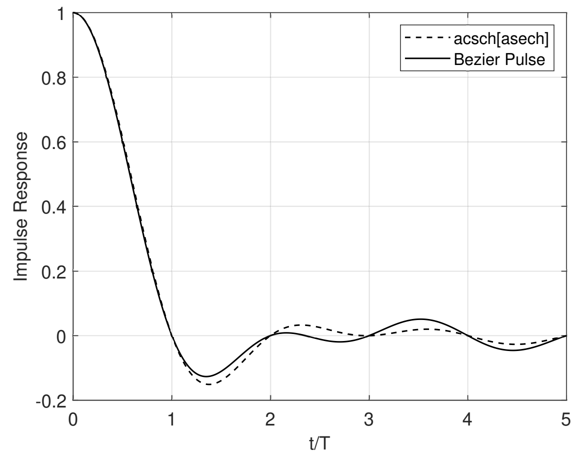

Fig. 4. Impulse responses of the considered Nyquist pulses for roll-off factor $\alpha = 0.25$.

perior to existing state-of-the-art designs (Table I) across all practical combinations of roll-off factors α and timing jitter τ. To ensure the proposed pulse strictly outperforms the benchmarks in every scenario, the determination of the control point coordinates (x_1, y_1) is formulated as a minimax optimization problem.

Specifically, instead of minimizing the average error, the objective is to minimize the worst-case performance ratio relative to the benchmark. The objective function $J(x_1, y_1)$ is defined as:

$$J(x_1, y_1) = \min_{(x_1,y_1)\in\mathbb{R}^2} \left[\max_{i,j} \left(\frac{\mathrm{BER}_{\text{Bézier}}(\alpha_i, \tau_j \mid x_1, y_1)}{\mathrm{BER}_{\mathrm{Ref}}(\alpha_i, \tau_j)} \right) \right], \tag{6}$$

where $\mathrm{BER}_{\mathrm{Ref}}$ represents the state-of-the-art benchmark BER values listed in Table I. A value of $J < 1$ guarantees that the proposed pulse achieves a lower bit error rate than the benchmark for all tested combinations. The optimization was performed using Particle Swarm Optimization (PSO) in MATLAB. PSO was selected for its global search capabilities, ensuring the solution avoids local minima inherent in the complex BER landscape. The optimization evaluated the pulse over the set of roll-off factors $\alpha_i \in \{0.2, 0.25, 0.35, 0.5\}$ and timing jitter values $\tau_i \in \{0.05, 0.1, 0.2, 0.3\}$ normalized to the symbol period T. The procedure converged to the optimal control point coordinates $p_1^{\mathrm{opt}} = (-0.5935, 2.0001)$.

Fig. 3 and Fig. 4 present the frequency response and impulse response, respectively, of the proposed Optimal Bézier Pulse (OBP). For comparative purposes, the acsch[asech] pulse [6] is also included. Notably, the frequency response of the proposed OBP is discontinuous, a characteristic that emerges directly from the pulse definition in (4) and the unconstrained optimization approach employed.

Table I provides a comprehensive comparison of the BER performance of the proposed OBP with existing state-of-the-art pulses under various roll-off factors and timing jitter conditions. The numerical results were obtained assuming a signal-to-noise ratio of 15 dB, $N = 2^9$ interfering symbols, a series expansion order of $N_m = 23$, a scaling parameter of $T = 37.5$, and an interfering symbol range of $[N_1, N_2] = [-50, 50]$. Across all evaluated scenarios, the proposed OBP consistently achieves the lowest BER, highlighting its improved robustness to timing synchronization errors.

III. Conclusion

This paper has presented a novel methodology for designing ISI-free Nyquist pulses based on quadratic Bézier curves. By representing the transition band of the frequency response as a parametric curve, we have established a flexible design framework parameterized by the coordinates of a single control point. Through systematic unconstrained optimization, we have identified the optimal control point coordinates that minimize the average BER for binary antipodal signaling under various timing jitter conditions.

The proposed Optimal Bézier Pulse (OBP) demonstrates superior error performance compared to established benchmarks, including the raised-cosine pulse and other recently proposed parametric pulse designs, particularly in scenarios characterized by significant timing jitter. The numerical results confirm that the OBP achieves consistent BER improvements across a wide range of roll-off factors and timing offset values.

Consequently, the proposed OBP constitutes a robust alternative for enhancing the reliability of digital communication systems operating under imperfect synchronization conditions. Future work may explore extensions to higher-order Bézier curves and their potential for further performance optimization.

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